



Cokernel Fuzzy Filters and a Fuzzy Congruence Induced by a Fuzzy Filters in MS-Algebras

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Abstract

In this paper, we studied Cokernel fuzzy filters of MS- algebras and fuzzy congruence relation generated by fuzzy filters and its properties. We also proved that the fuzzy congruence relation generated by fuzzy filters is the superimum of a lattice fuzzy congruence generated by fuzzy filter μ and a lattice a fuzzy congruence generated by fuzzy ideal μ_α . Finally, we proved that the set of fuzzy congruence cokernels of L , ordered by set inclusion, forms a complete lattice.

Keywords: Cokernel fuzzy filters, MS-algebras, Fuzzy congruencies

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1. Introduction

Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of all Ockham algebras contains the well-known classes, forexamples Boolean algebras, de Morgan algebras, Kleene algebras and Stone algebras [6]. Blyth and Varlet [6, 7] defined a subclass of Ockham algebras so called MS-algebras which generalize both de Morgan algebras and Stone algebras. These algebras belong to the class of Ockham algebras introduced by Berman [5]. The class of all MS-algebras forms an equation class. Blyth and Varlet [8] characterized the sub varieties of MS-algebras.

On the other way, fuzzy set theory introduced by Zadeh [14] is generalization of classical set theory. Next, Rosenfeld [11] applied it to group theory and developed the theory of fuzzy subgroups. Also, many authors have worked on fuzzy lattice theory. They introduced the concepts of fuzzy sub lattice; fuzzy ideal, fuzzy prime ideal, in a lattice gave some interesting results (see [3, 4, 9, 12, 13]). More recently, Alaba and Alemayehu [1] studied kernel fuzzy ideals and a fuzzy congruence in MS-algebras. Alaba and Alemayehu [2] introduced fuzzy ideals in demi-pseudo complemented MS-algebras. In this paper, cokernel fuzzy filters of MS-algebras were introduced and characterized.

1. Preliminaries

In this section, we discuss some definitions and results which used be used in this paper.

Definition: 2.1. [6, 7] An MS-algebra is an algebra $(L, \vee, \wedge, \circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$, such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \rightarrow a^\circ$ is a unary operation satisfies: $a \leq a^{\circ\circ}$, $(a \wedge b)^\circ = a^\circ \vee b^\circ$, $1^\circ = 0$

Lemma: 2.2. [6, 7] For any two elements a, b of an MS-algebra, we have the following:

- (1) $0^\circ = 1$
- (2) $a \leq b \Rightarrow b^\circ \leq a^\circ$
- (3) $a^{\circ\circ} = a^\circ$
- (4) $(a \vee b)^\circ = a^\circ \wedge b^\circ$
- (5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$
- (6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$

Definition: 2.3. [14] Let μ be a fuzzy subset of S and let $\alpha \in [0, 1]$. Then the set $\mu_\alpha = \{x \in L : \alpha \leq \mu(x)\}$ is called a level subset of μ .

Definition: 2.4. [12] A fuzzy subset μ of a bounded lattice L is said to be a fuzzy ideal of L , if for all $x, y \in L$,

- (1) $\mu(0) = 1$,
- (2) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$
- (3) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$ for all $x, y \in L$.

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Definition: 2.5. [12] A fuzzy subset μ of a bounded lattice L is said to be a fuzzy filter of L , if for all $x, y \in L$,

- (1) $\mu(1) = 1$,
- (2) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$
- (3) $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$ for all $x, y \in L$.

In [12], Swamy and Raju observed that, a fuzzy subset μ of a bounded lattice L is a fuzzy ideal of L if and only if $\mu(0) = 1$ and $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ for all $x, y \in L$.

A fuzzy subset μ of a bounded lattice L is a fuzzy filter of L if and only if $\mu(1) = 1$ and $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ for all $x, y \in L$.

A fuzzy relation θ on a set X is map $\theta: X \times X \rightarrow [0, 1]$. For any $x, y \in X$ and fuzzy relations θ and φ on X , $(\theta \cap \varphi)(x, y) = \min\{\theta(x, y), \varphi(x, y)\}$, $(\theta \cup \varphi)(x, y) = \max\{\theta(x, y), \varphi(x, y)\}$, $\theta \subseteq \varphi$ means $\theta(x, y) \leq \varphi(x, y)$.

Definition: 2.6. [10] Suppose that θ and φ are two fuzzy relations on a set X . Then $(\theta \circ \varphi)(x, y) = \sup_{z \in X} (\theta(x, z) \wedge \varphi(z, y))$.

Definition: 2.7. [10] A fuzzy relation φ on X is said to be a fuzzy equivalence relation on X , if

- (1) $\varphi(x, x) = 1$ for all $x \in X$ (reflexive),
- (2) $\varphi(x, y) = \varphi(y, x)$ for all $x, y \in L$ (symmetric),
- (3) $\varphi(x, z) \geq \varphi(x, y) \wedge \varphi(y, z)$ for all $x, y, z \in L$ (transitive).

Definition: 2.8. [1] A fuzzy relation φ on an MS-algebra L is called fuzzy congruence relation on L if the following are satisfied:

- (1) $\varphi(x \wedge z, y \wedge w) \wedge \varphi(x \vee z, y \vee w) \geq \varphi(x, y) \wedge \varphi(z, w)$ for all $x, y, z, w \in L$,
- (2) $\varphi(x^\circ, y^\circ) \geq \varphi(x, y)$ for all $x, y \in L$.

Definition: 2.9. [1] A kernel fuzzy ideal μ of an MS-algebra L is a fuzzy ideal μ of L for which there exists a fuzzy congruence φ of L such that $\mu = \ker \varphi$ i.e μ is the fuzzy kernel ideal of φ , where $\mu(x) = \ker \varphi(x) = \varphi(x, 0)$.

Throughout the next sections, L stands for an MS-algebra unless another word is mentioned.

2. Cokerner Fuzzy Filters And a fuzzy congruence induced by a fuzzy filters in MS-algebras

In this Section, cokernel fuzzy filters of MS-algebras are introduced and characterized.

Let L be an MS-algebras L and μ be a fuzzy filter of L . Define

$$\mu_\circ(x) = \sup\{\mu(f) : x \leq f^\circ, \text{ for some } f \in L\}$$

$$\mu^{\circ\circ}(x) = \sup\{\mu(f) : f^{\circ\circ} \leq x, \text{ for some } f \in L\}.$$

Then, we have the following results.

Lemma 3.1. μ_\circ is a fuzzy ideal of L and $\mu^{\circ\circ}$ is a fuzzy filter of L .

Proof. Clearly $\mu_\circ(0) = 1$. For any $x, y \in L$,

$$\mu_\circ(x) \wedge \mu_\circ(y) = \sup\{\mu(f_1) : x \leq f_1^\circ, \text{ for some } f_1 \in L\} \wedge \sup\{\mu(f_2) : y \leq f_2^\circ, \text{ for some } f_2 \in L\}$$

$$= \sup\{\mu(f_1) \wedge \mu(f_2) : x \leq f_1^\circ, y \leq f_2^\circ \text{ for some } f_1, f_2 \in L\}$$

$$\leq \sup\{\mu(f_1 \wedge f_2) : x \vee y \leq (f_1 \wedge f_2)^\circ \text{ for some } f_1, f_2 \in L\}$$

$$= \mu_\circ(x \vee y).$$

$$\mu_\circ(x) \vee \mu_\circ(y) = \sup\{\mu(f_1) : x \leq f_1^\circ, \text{ for some } f_1 \in L\} \vee \sup\{\mu(f_2) : y \leq f_2^\circ, \text{ for some } f_2 \in L\}$$

$$= \sup\{\mu(f_1) \vee \mu(f_2) : x \leq f_1^\circ, y \leq f_2^\circ \text{ for some } f_1, f_2 \in L\}$$

$$\leq \sup\{\mu(f_1 \vee f_2) : x \wedge y \leq (f_1 \vee f_2)^\circ \text{ for some } f_1, f_2 \in L\}$$

$$= \mu_\circ(x \wedge y)$$

Hence μ_\circ is a fuzzy ideal of L .

Similarly, we prove that $\mu^{\circ\circ}$ is a fuzzy filter of L .

Lemma: 3.2. For any two fuzzy filters μ and λ of an MS-Algebra L , then the following hold.

- (1) $\mu^{\circ\circ} \subseteq \mu$,
- (2) If $\mu \subseteq \lambda$, then $\mu^{\circ\circ} \subseteq \lambda^{\circ\circ}$
- (3) $(\mu \cap \lambda)^{\circ\circ} = \mu^{\circ\circ} \cap \lambda^{\circ\circ}$,
- (4) $(\mu \vee \lambda)^{\circ\circ} = \mu^{\circ\circ} \vee \lambda^{\circ\circ}$.

Proof: (1) $\mu^{\circ\circ}(x) = \sup\{\mu(f) : f \leq f^{\circ\circ} \leq x, \text{ for some } f \in L\} \leq \mu(x)$ as μ is a fuzzy filter of L . Hence, $\mu^{\circ\circ} \subseteq \mu$.

(2) The proof of (2) is straightforward.

(3) By (2) $(\mu \cap \lambda)^{\circ\circ} \subseteq \mu^{\circ\circ} \cap \lambda^{\circ\circ}$. For any $x \in L$, $(\mu^{\circ\circ} \cap \lambda^{\circ\circ})(x) = \mu^{\circ\circ}(x) \wedge \lambda^{\circ\circ}(x)$

$$= \sup\{\lambda(f_1) : f_1^{\circ\circ} \leq x\} \wedge \sup\{\mu(f_2) : f_2^{\circ\circ} \leq x\}$$

$$= \sup\{\lambda(f_1) \wedge \mu(f_2) : f_1^{\circ\circ} \leq x, f_2^{\circ\circ} \leq x\}$$

$$\leq \sup\{\lambda(f_1 \vee f_2) \wedge \mu(f_1 \vee f_2) : (f_1 \vee f_2)^{\circ\circ} \leq x\}$$

$$\leq \sup\{(\mu \cap \lambda)(f_1 \vee f_2) : (f_1 \vee f_2)^{\circ\circ} \leq x\}$$

$$= (\mu \cap \lambda)^{\circ\circ}(x).$$

This implies $(\mu \cap \lambda)^{\circ\circ} = \mu^{\circ\circ} \cap \lambda^{\circ\circ}$

(4) By (2) $\mu^{\circ\circ} \vee \lambda^{\circ\circ} \subseteq (\mu \vee \lambda)^{\circ\circ}$.

Conversely, let $x \in L$,

$$\begin{aligned}
 (\mu^{\circ\circ} \vee \lambda^{\circ\circ})(x) &= \sup\{\mu^{\circ\circ}(x_1) \wedge \lambda^{\circ\circ}(x_2) : x = x_1 \wedge x_2\} \\
 &= \sup\{\sup\{\mu(a_1) : a_1^{\circ\circ} \leq x_1\} \wedge \sup\{\lambda(a_2) : a_2^{\circ\circ} \leq x_2\} : x_1 \wedge x_2 = x\} \\
 &= \sup\{\sup\{\mu(a_1) \wedge \lambda(a_2) : a_1^{\circ\circ} \leq x_1, a_2^{\circ\circ} \leq x_2\} : x = x_1 \wedge x_2\} \\
 &= \sup\{\mu(a_1) \wedge \lambda(a_2) : a_1^{\circ\circ} \leq x_1, a_2^{\circ\circ} \leq x_2, x = x_1 \wedge x_2\}
 \end{aligned}$$

Put $A = \{\alpha \in [0, 1], \text{ where } \alpha = \mu(a_1) \wedge \lambda(a_2) : a_1^{\circ\circ} \leq x_1, a_2^{\circ\circ} \leq x_2, x = x_1 \wedge x_2\}$. Again,

$$\begin{aligned}
 (\mu \vee \lambda)^{\circ\circ}(x) &= \sup\{(\mu \vee \theta)(a) : a^{\circ\circ} \leq x\} \\
 &= \sup\{\sup\{\mu(a_1) \wedge \theta(a_2) : a_1 \wedge a_2 = a\} : a^{\circ\circ} \leq x\} \\
 &= \sup\{\mu(a_1) \wedge \lambda(a_2) : a_1 \wedge a_2 = a, a^{\circ\circ} \leq x\}
 \end{aligned}$$

Put $B = \{\gamma \in [0, 1], \text{ where } \gamma = \mu(a_1) \wedge \lambda(a_2) : a_1 \wedge a_2 = a, a^{\circ\circ} \leq x\}$. Now we prove that $B \subseteq A$.

Let $\gamma \in B$. Then $\gamma = \mu(a_1) \wedge \theta(a_2)$ such that $a_1 \wedge a_2 = a, x \geq a^{\circ\circ}$.

This implies $x \geq (a_1 \wedge a_2)^{\circ\circ} = a_1^{\circ} \wedge a_2^{\circ}$.

This implies $x = x \vee (a_1^{\circ} \wedge a_2^{\circ}) = (x \vee a_1^{\circ}) \wedge (x \vee a_2^{\circ})$.

Put $x_1 = x \vee a_1^{\circ}, x_2 = x \vee a_2^{\circ}$. This implies $x_1 \geq a_1^{\circ}$, and $x_2 \geq a_2^{\circ}$, where $x = x_1 \wedge x_2$. This implies $\gamma \in A$ and $B \subseteq A$.

This implies $\sup B \leq \sup A$. Then we get $(\mu \vee \mu)^{\circ\circ} = \mu_{\circ\circ} \vee \lambda_{\circ\circ}$.

(2) Similarly (2) holds.

Corollary: 3.3. If $\{\mu_\alpha : \alpha \in \Delta\}$ is a family of filters of L , then we have the following:

- (1) $(\bigcap_{\alpha \in \Delta} \mu_\alpha)^{\circ\circ} = \bigcap_{\alpha \in \Delta} (\mu_\alpha)^{\circ\circ}$
- (2) $(\bigvee_{\alpha \in \Delta} \mu_\alpha)^{\circ\circ} = \bigvee_{\alpha \in \Delta} (\mu_\alpha)^{\circ\circ}$.

Lemma: 3.4. Let μ be a fuzzy filter of L . Then the following hold:

- (1) $\mu^{\circ\circ}(x^{\circ}) \geq \mu_{\circ}(x)$
- (2) $\mu_{\circ}(x^{\circ}) \geq \mu^{\circ\circ}(x)$

Proof: (1) Let $x \in L$.

$$\begin{aligned}
 \mu_{\circ}(x) &= \sup\{\mu(f) : x \leq f^{\circ}, \text{ for some } f \in L\} \\
 &\leq \sup\{\mu(f) : f^{\circ\circ} \leq x^{\circ}, \text{ for some } f \in L\} \\
 &= \mu^{\circ\circ}(x^{\circ})
 \end{aligned}$$

We recall that the fuzzy congruence class of a non-empty set L determined by x and a fuzzy congruence φ , denoted by φ_x , is the fuzzy subset of L defined by $\varphi_x(y) = \varphi(x, y), \forall y \in L$. Let L/φ denote the set of all fuzzy congruence class, that is $L/\varphi = \{\varphi_x : x \in L\}$.

If φ is a fuzzy congruence of L and $x, y \in L$, then $\varphi_x = \varphi_y \Leftrightarrow \varphi(x, y) = 1$.

Definition:3.5. A fuzzy filter μ of L is said to be a cokernel fuzzy filter if and only if there is a fuzzy congruence relation φ of L such that $\mu = \varphi[1]$.

Definition 3.6. Let L be an MS-algebra and μ a fuzzy filter of L . Define a fuzzy binary relation $\varphi(\mu)$ on L as follows: $\varphi(\mu)(x, y) = \sup\{\mu(f) \wedge \mu_{\circ}(t) : (x \wedge f) \vee t = (y \wedge f) \vee t \text{ for some } f, t \in L\}$.

Then we have the following results:

Theorem 3.7. $\varphi(\mu)$ is a fuzzy congruence relation on L .

Proof. Clearly $\varphi(\mu)$ is reflexive and symmetric. We first show that $\varphi(\mu)$ is transitive.

Consider $x, y, z, s, r, k \in L$ such that

$$\begin{aligned}
 (x \wedge s) \vee t &= (y \wedge s) \vee t \text{ and} \\
 (y \wedge r) \vee k &= (z \wedge r) \vee k
 \end{aligned}$$

Put $a = s \wedge r$ and $b = t \vee k$

$$(x \wedge a) \vee b = (x \wedge s \wedge r) \vee t \vee k$$

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$$\begin{aligned}
 &= ((x \wedge s) \vee (t \vee k) \wedge (r \vee t \vee k)) \\
 &= (((x \wedge s) \vee t) \vee k) \wedge (r \vee t \vee k) \\
 &= (((y \wedge s) \vee t) \vee k) \wedge (r \vee t \vee k) \\
 &= \{((x \wedge s) \vee t) \vee (r \wedge t)\} \wedge k \\
 &= ((y \wedge r) \vee k) \vee t \wedge (s \vee t \vee k) \\
 &= ((z \wedge r) \vee k) \vee t \wedge (s \vee t \vee k) \\
 &= (((z \wedge s \wedge r) \vee (t \vee k)) \\
 &= ((z \wedge a) \vee b)
 \end{aligned}$$

Now, we have the following:

$$\begin{aligned}
 \varphi(\mu)(x, y) \wedge \varphi(\mu)(y, z) &= \sup\{\mu(s) \wedge \mu_o(t) : (x \wedge s) \vee t = (y \wedge s) \vee t\} \wedge \\
 &\quad \sup\{\mu(s) \wedge \mu_o(t) : (y \wedge r) \vee k = (z \wedge r) \vee k\} \\
 &= \sup\{\mu(s) \wedge \mu_o(t) \wedge \mu(s) \wedge \mu_o(t) : (x \wedge s) \vee t = (y \wedge s) \vee t, \\
 &\quad (y \wedge r) \vee k = (z \wedge r) \vee k\} \\
 &\leq \sup\{\mu(s \vee r) \wedge \mu_o(t \wedge k) : (x \wedge a) \vee b = (y \wedge a) \vee b\} \\
 &= \varphi(\mu)(x, y)
 \end{aligned}$$

Hence $\varphi(\mu)$ is transitive.

Therefore, it is an equivalence relation. By following, the same procedure, we can show that $\varphi(\mu)$ satisfies the substitution property for \vee and \wedge . It remains to show that $\varphi(\mu)$ satisfies the substitution property for the unary operation \circ . Let $(x, y) \in L$. Then

$$\begin{aligned}
 \varphi(\mu)(x, y) &= \sup\{\mu(s) \wedge \mu_o(t) : (x \wedge s) \vee t = (y \wedge s) \vee t\} \\
 &\leq \sup\{\mu(t^\circ) \wedge \mu_o(s^\circ \wedge t^\circ) : (x^\circ \wedge t^\circ) \vee (s^\circ \wedge t^\circ) = (y^\circ \wedge t^\circ) \vee (s^\circ \wedge t^\circ)\} \\
 &= \varphi(\mu)(x^\circ, y^\circ)
 \end{aligned}$$

Thus, $\varphi(\mu)$ is a congruence relation.

Theorem 3.8. For each fuzzy filter μ of L , $\varphi(\mu) = \varphi_{lat}(\mu) \vee \varphi_{lat}(\mu_o)$, where the join is taken in the lattice of lattice fuzzy congruences on L .

Proof. First we show that $\varphi(\mu)$

$$\varphi_{lat}(\mu) \vee \varphi_{lat}(\mu_o) \subseteq \varphi(\mu)$$

Clearly $\varphi_{lat}(\mu) \subseteq \varphi(\mu)$. Next, we prove that $\varphi_{lat}(\mu_o) \subseteq \varphi(\mu)$. Suppose that $x, y \in L$ such that

$$a \vee x = a \vee y$$

for some $a \in L$. Then we have

$$\begin{aligned}
 (a^\circ \wedge a) \vee (a^\circ \wedge x) &= a^\circ \wedge (a \vee x) \\
 &= a^\circ \wedge (a \vee y) \\
 &= (a^\circ \wedge a) \vee (a^\circ \wedge y)
 \end{aligned}$$

$$\begin{aligned}
 \varphi_{lat}(\mu_o)(x, y) &= \sup\{\mu_o(a) : x \vee a = y \vee a\} \\
 &\leq \sup\{\mu^{\circ\circ}(a) \wedge \mu_o(a \wedge a^\circ) : (a^\circ \wedge a) \vee (a^\circ \wedge x) = (a^\circ \wedge a) \vee (a^\circ \wedge y)\} \\
 &\leq \sup\{\mu(a^\circ) \wedge \mu_o(a \wedge a^\circ) : (a^\circ \wedge a) \vee (a^\circ \wedge x) = (a^\circ \wedge a) \vee (a^\circ \wedge y)\} \\
 &\quad \text{as } \mu^{\circ\circ} \subseteq \mu \\
 &= \varphi(\mu)(x, y)
 \end{aligned}$$

Hence, $\varphi_{lat}(\mu_o) \vee \varphi_{lat}(\mu) \subseteq \varphi(\mu)$.

Next, we show that $\varphi(\mu) \subseteq (\varphi_{lat}(\mu_o) \vee (\varphi_{lat}(\mu)^\circ))$.

$$\begin{aligned}
 \varphi(\mu)(x, y) &= \sup\{\mu_o(s) \wedge \mu^{\circ\circ}(t) : (x \wedge s) \vee t = (y \wedge s) \vee t, s, t \in L\} \\
 &\leq \sup\{\mu_o(s) \wedge \mu^{\circ\circ}(t) : x \vee t = x \vee (x \wedge s) \vee t = x \vee (y \wedge s) \vee t, i, j \in L\} \\
 &\leq \sup\{\mu^{\circ\circ}(t) : x \vee t = x \vee (x \wedge s) \vee t = x \vee (y \wedge s) \vee t, s, t \in L\} \\
 &= (\varphi_{lat}(\mu)^\circ)(x, x \vee (y \wedge s)) \\
 \varphi(\mu)(x, y) &= \sup\{\mu_o(s) \wedge \mu^{\circ\circ}(t) : (x \wedge s) \vee t = (y \wedge s) \vee t, s, t \in L\} \\
 &\leq \sup\{\mu_o(s) \wedge \mu^{\circ\circ}(t) : x \vee t = x \wedge (x \wedge s) \vee t = x \vee (y \wedge s) \vee t, s, t \in L\} \\
 &\leq \sup\{\mu_o(s) \wedge \mu^{\circ\circ}(t) : (x \vee t) \wedge s = (x \vee (x \wedge s) \vee t) \wedge s
 \end{aligned}$$

$$\begin{aligned}
 &= (x \vee (y \wedge s) \vee t) \wedge s, \quad s, t \in L \\
 &\leq \sup\{\mu_{\circ\circ}(s) : (x \vee (y \wedge s)) \wedge s = (x \wedge s) \vee ((y \wedge s) \wedge s)\} \\
 &= (x \vee y) \wedge s, \quad s, t \in L \\
 &= (\varphi_{lat})(\mu_{\circ\circ})(x \vee (y \wedge s), x \vee y)
 \end{aligned}$$

This implies,

$$\begin{aligned}
 \varphi(\mu)(x, y) &\leq (\varphi_{lat})(\mu^{\circ})(x, x \vee (y \wedge s)) \wedge (\varphi_{lat})(\mu_{\circ\circ})(x \vee (y \wedge s), x \vee y) \\
 &\leq ((\varphi_{lat})(\mu^{\circ}) \vee (\varphi_{lat})(\mu_{\circ\circ}))(x, x \vee y)
 \end{aligned}$$

Similarly, $\Omega_L(\mu)(x, y) \leq ((\Omega_{lat})_L(\mu^{\circ}) \vee (\Omega_{lat})_L(\mu_{\circ\circ}))(x \wedge y, y)$. Again,

$$\begin{aligned}
 \varphi(\mu)(x, y) &\leq ((\varphi_{lat})(\mu^{\circ}) \vee (\varphi_{lat})(\mu_{\circ\circ}))(x, x \wedge y) \vee ((\varphi_{lat})(\mu^{\circ}) \vee (\Omega_{lat})(\mu_{\circ\circ}))(x \vee y, y) \\
 &\leq ((\varphi_{lat})(\mu^{\circ}) \vee (\Omega_{lat})(\mu_{\circ\circ}))(x, y)
 \end{aligned}$$

$$\text{Thus, } \varphi(\mu) = ((\varphi_{lat})(\mu^{\circ}) \vee (\Omega_{lat})(\mu_{\circ\circ}))$$

Corollary 3.9. For each fuzzy filter μ of L , $\varphi(\mu)$ is the smallest fuzzy congruence containing $\mu \times \mu$

Theorem 3.10. A fuzzy filter μ of an MS-algebras L is a cokernel fuzzy filter if and only if $\mu_{\circ}(j) \wedge \mu(x \vee j) \leq \mu(x)$.

Proof. Suppose that μ is a cokernel fuzzy filter, then $\mu = \text{coker}\phi$ for some fuzzy congruence of L .

$$\begin{aligned}
 \mu_{\circ}(j) \wedge \mu(x \vee j) &= \sup\{\mu(t) : j \leq t^{\circ}, \text{ for some } t \in L\} \wedge \mu(x \vee j) \\
 &= \sup\{\text{coker}\phi(t) : j \leq t^{\circ}, \text{ for some } t \in L\} \wedge \text{coker}\phi(x \vee j) \\
 &\leq \phi(t^{\circ}, 0) \wedge \phi(x \vee j, 1) \\
 &\leq \phi(j, 0) \wedge \phi(x \vee j, 1) \quad \text{as } j \leq t^{\circ} \\
 &\leq \phi(x \vee j, x) \wedge \phi(x \vee j, 1) \\
 &\leq \phi(1, x) \\
 &= \text{coker}\phi(x) \\
 &= \mu(x)
 \end{aligned}$$

Hence, $\mu_{\circ}(j) \wedge \mu(x \vee j) \leq \mu(x)$

Conversely, suppose that $\mu_{\circ}(j) \wedge \mu(x \vee j) \leq \mu(x)$ holds.

Clearly $\mu \subseteq \text{coker}\phi(\mu)$.

$$\begin{aligned}
 \text{coker}\phi(\mu)(x) &= \varphi(\mu)(x, 1) \\
 &= \sup\{\mu(s) \wedge \mu_{\circ}(t) : (x \wedge s) \vee t = s \vee t, \quad s, t \in L\} \\
 &\leq \sup\{\mu(s \vee t) \wedge \mu_{\circ}(t) : (x \vee s) \wedge t = s \wedge t, \quad s, t \in L\} \\
 &= \sup\{\mu((x \wedge s) \vee t) \wedge \mu_{\circ}(t) : (x \vee s) \wedge t = s \wedge t, \quad s, t \in L\} \\
 &\leq \mu(x \wedge s) \text{ by hypothesis} \\
 &\leq \mu(x)
 \end{aligned}$$

Hence, $\text{coker}\phi(\mu) \subseteq \mu$. Thus $\text{coker}\phi(\mu) = \mu$

Theorem 3.11. A prime fuzzy filter μ of an MS-algebras L is a fuzzy congruence cokernel if and only if $(\mu \cap \mu_{\circ})(x) \leq a$ for some $a \in \text{Im}(\mu)$

Theorem 3.12. Let μ be a fuzzy filter of an MS-algebras L , and ψ is a fuzzy congruence of L . If $\text{coker}\psi = \mu$, then $\mu_{\circ} = \text{ker}\psi$. Moreover, if $\text{coker}\psi$ is prime fuzzy filter, then $\text{ker}\psi$ is a prime fuzzy ideal of L .

The set of all fuzzy congruence cokernels of L , denoted by $\text{FF}_{CK}(L)$.

Lemma 3.13. Let $\{\mu_i : i \in \Omega\}$ be a family of fuzzy congruence cokernels of L . Then $\bigcap_{i \in \Omega} \mu_i$ is fuzzy congruence cokernels of L .

Theorem 3.14. The set $\text{FF}_{CK}(L)$ of L -fuzzy congruence cokernels of L , ordered by set inclusion, forms a complete lattice.

Data availability

No data were used to support this study

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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