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A review on phase plane and non-linear system of differential equations with application

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Abstract

In this paper we review on nonlinear phenomena and properties, particularly those with physical relevance. Often, mathematical models of real-world phenomena are formulated in terms of systems of nonlinear differential equations, which may be difficult to solve explicitly. Finding a solution to a differential equation may not be so important if that solution never appears in the physical model represented by the system, or is only realized in exceptional circumstances. Thus, equilibrium solutions, which correspond to configurations in which the physical system does not move, only occur in everyday situations if they are stable. An unstable equilibrium will not appear in practice, since slight perturbations in the system or its physical surroundings will immediately dislodge the system far away from equilibrium. We take a qualitative approach to the analysis of solutions to nonlinear systems by making phase portraits and using stability analysis.

Keywords: Phase portrait, Phase trajectory, Closed orbit, Separatrix, Eigenvalues, Linearization

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Introduction

Differential equations first came into existence with the invention of calculus by physicist Sir Isaac Newton (1642 – 1727) and the mathematician Gottfried Wilhelm Leibniz (1646 – 1716) (Newton, 1744). In mathematics, a differential equation is an equation that relates one or more unknown functions and their derivatives (Zill, 2012). In applications, the functions generally represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. Such relations are common; therefore, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology. Differential equations can be divided into several types. Commonly used distinctions include whether the equation is ordinary or partial, linear or non-linear, and homogeneous or heterogeneous. An ordinary differential equation (ODE) is an equation containing an unknown function of one real or complex variable y, its derivatives, and some given functions of x. The unknown function is generally represented by a variable (often denoted y), which, therefore, depends on x. Thus x is often called the independent variable of the equation. The term "ordinary" is used in contrast with the term partial. A partial differential equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs can be used to describe a wide variety of phenomena in nature such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics. Linear differential equations are the differential equations that are linear in the unknown function and its derivatives. A non-linear differential equation is a differential equation that is not a linear equation in the unknown function and its derivatives. Non-linear differential equations can exhibit very complicated behavior over extended time intervals, characteristic of chaos (complete disorder and confusion) (Byatt-Smith, 1979 and Strogatz, 2018).

Linear differential equations frequently appear as approximations to nonlinear equations. These approximations are only valid under restricted conditions. For example, the harmonic oscillator equation is an approximation to the nonlinear pendulum equation that is valid for small amplitude oscillations. Differential equations play an important role in modeling virtually every physical, technical, or biological process; from celestial motion to bridge design, to interactions between neurons (Johnson, E., and Johnson, J. R. 1965). Another characteristic of differential equation in the 20th century was the creation of geometrical or topological methods, especially for non-Strogatz, S. H. (2018). Linear DEs (Strogatz, S. H. 2018). The goal is to understand at least the qualitative behavior of solutions from a geometrical, as well as from an analytical point of view.

Natural systems are highly non-linear. Interactions between individuals, species, or populations lead to relationships that depend on variables in a way more complicated than that of simple proportionality. Among other things, this means that models describing such phenomena contain non-linear equations that are often difficult, if not impossible, to solve explicitly in closed form. Our aim is to solve the limitation for calculating explicitly solutions to non-linear ODE's. We would be focusing on determining qualitative features of these solutions. The approach is graphical and geometric; and provides a description of the solutions behavior, which allow us to understand the phenomena captured in the modeling in a pictorial form.

Non-linear ordinary differential equation play an important role in many branches of applied and pure mathematics and their applications in engineering ,applied mechanics ,quantum physics ,analytical chemistry ,astronomy and biology (Strogatz, 2018). From last decades, researchers pay attentions towards analytical and numerical solutions of non-linear ordinary differential equations. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving non- linear ordinary differential equations. Realizing that non-linear differential is rarely solvable analytically and not yet having the benefit of computers generate solution numerically.The theory and applications of differential equations and their system play an important role in modern dynamics. Such equations are mathematical models of various real-life physical phenomena.

Materials and Methods

The study of differential equations is a wide field in pure and applied mathematics, physics, and engineering (Bender, Orszag, S., and Orszag, S. A. 1999). All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behavior of complex systems.

Important preliminaries

Before analyzing systems of ordinary differential equations, we better first establish the existence of solution of an ordinary differential equation which is fundamental for further analyze. This is the purpose of the existence and uniqueness theory for systems of differential equations.

*Theorem***: (**Existence and uniqueness of solution**)**

Let R be a rectangular region in the $x, y-$plane defined by $a \le x \le b \text{ and } c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y) \text{ and } \frac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0: (x_0 - h, x_0 + h)$, h>0, contained in $[a, b]$, and a unique function $y(x)$, defined in I_0, that is a solution of initial value problem.

Definition: In mathematics, an autonomous system or autonomous differential equation is a system of ordinary differential equations which does not explicitly depend on the independent variable. When the variable is time, they are also called time-invariant systems. It is the form of $\frac{dy}{dt} = f(y)$.

Definitions:- In applied mathematics, in particular the context of nonlinear system analysis, a phase plane is a visual display of certain characteristics of certain kinds of differential equations; a coordinate plane with axes being the values of the two state variables, say (x, y), or (any pair of variables). It is a two-dimensional case of the general n-dimensional phase space.

Definition:- A phase portrait or integral curve is a geometric representation of the trajectories of a dynamical system in the phase plane. Each set of initial conditions is represented by a different curve, or point.

Definition:- Let $\begin{cases} \frac{dx}{dt} = f(x,\ y) \\ \frac{dy}{dt} = g(x,\ y) \end{cases}$ be a system of autonomous ordinary differential equation. We say that any point (x_0, y_o) which both f $(x,\ y)$ and g $(x,\ y)$ vanish that is; $f(x_0, y_0) = g(x_0, y_0) = 0$ is called a singular point (or critical point).

From a dynamical point of view a singular point is an equilibrium point, or fixed point, because if we start at such appoint then we remain there for all time t because $\frac{dx}{dt} = \frac{dy}{dt} = 0$. Singular points are of great importance in phase plane analysis.

To study a singular point of a non-linear system, we focused attention on its immediate neighborhood by expanding $f(x, y)$ and $g(x, y)$ in Taylor series about that point and linearizing; that is, cutting them off after the first –order terms. Thus, using Taylor's expansion series, $f(x, y)$ and $g(x, y)$ are approximated at equilibrium point (x_0, y_0) as:

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)[x - x_0] + \frac{\partial f}{\partial y}(x_0, y_0)[y - y_0]$$

$$g(x, y) \approx g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)[x - x_0] + \frac{\partial g}{\partial y}(x_0, y_0)[y - y_0]$$

But, at critical point $f(x_0, y_0) = g(x_0, y_0) = 0$. Hence, the non- linear autonomous system is reduced to:

$$\frac{dx}{dt} = f(x,\ y) = \frac{\partial f}{\partial x}(x_0, y_0)[x - x_0] + \frac{\partial f}{\partial y}(x_0, y_0)[y - y_0]$$
$$\frac{dy}{dt} = g(x,\ y\ ) = \frac{\partial g}{\partial x}(x_0, y_0)[x - x_0] + \frac{\partial g}{\partial y}(x_0, y_0)[y - y_0]$$

…….. … … …. . (θ)

This is the linearized system of non- linear system with coefficient matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

This matrix is called Jacobean matrix.

Therefore, (θ) is expressed by standard form as: $\frac{dX}{dt} = AX$, where $X = \begin{pmatrix} x \\ y \end{pmatrix}$ $and\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Stability and behavior of solutions of such the system can be determined using eigenvalues of A. The characteristic equation is $|A - \lambda I| = 0$

$$\Longrightarrow \lambda^2 - (a + d)\lambda + ad - bc = 0$$

$$\Longrightarrow \lambda^2 - (trA)\lambda + detA = 0$$

The solution are $\lambda_{1,2} = \frac{-trA \pm \sqrt{(trA)^2 - 4detA}}{2}$.

Based on the Eigen value of the solution of characteristic equation, we get the following cases:

Case 1. If $\lambda_{1,2}$ are both real, district and negative in sign, then the equilibrium point is stable node. This means that all solution curves of the two dimensional dynamical system attract towards the equilibrium point as shown Figure 2.

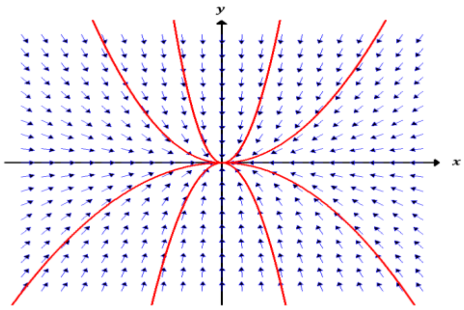

Figure 1. The graph of stable node to two dimensional dynamical system with (0,0) fixed point.

Case 2. If $\lambda_{1,2}$ are both real, district and positive in sign, then the fixed point is unstable node. This means that all solution curves of the two dimensional dynamical system go away from the equilibrium point as shown in Figure 1.

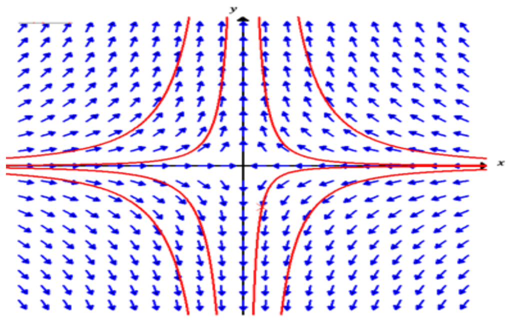

Figure 2. The graph of unstable node to two dimensional dynamical system with (0,0) fixed point.

Case 3. If $\lambda_{1,2}$ are both real, district and opposite in sign, then the equilibrium point is saddle node. This means that all solution curves of the two dimensional dynamical system asymptotic to the axes as shown in Figure 3.

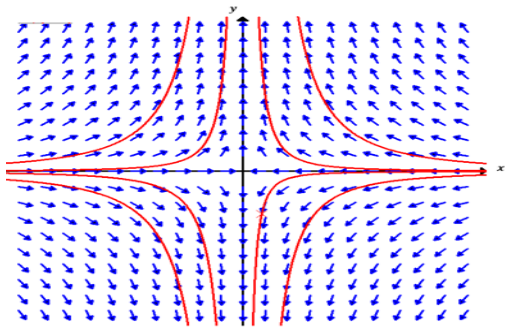

Figure 3. The graph of saddle node to two dimensional dynamical system with (0,0) fixed point.

Case 4. If $\lambda_{1,2}$ are complex, with real part negative then, the fixed point is stable spiral focus. This means that all solution curves of the two dimensional dynamical system spirally move towards the equilibrium point as shown in Figure 4.

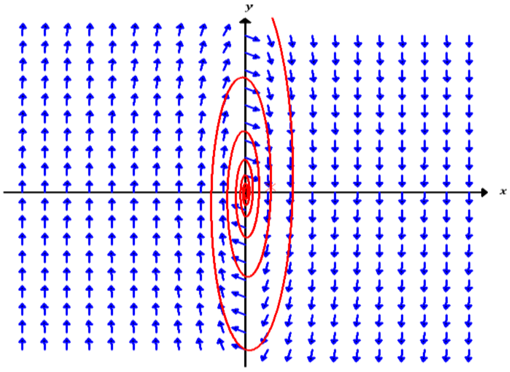

Figure 4. The graph of stable spiral focus to two dimensional dynamical system with (0,0) fixed point

Case 5. If $\lambda_{1,2}$ are complex, with real part positive then, the critical point is unstable spiral focus. This means that all solution curves of the two dimensional dynamical system spirally go away from the fixed point as indicated in Figure 5.

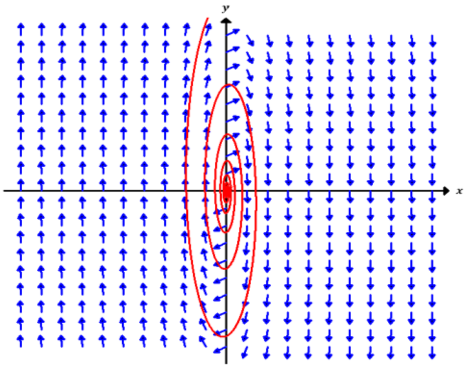

Figure 5. The graph of unstable spiral focus to two dimensional dynamical system with (0,0) fixed point

Case 6. If $\lambda_{1,2}$ are pure imaginary, then fixed point is center. This means that all solution curves of the two dimensional dynamical system makes a circle around the fixed point as shown in Figure 6.

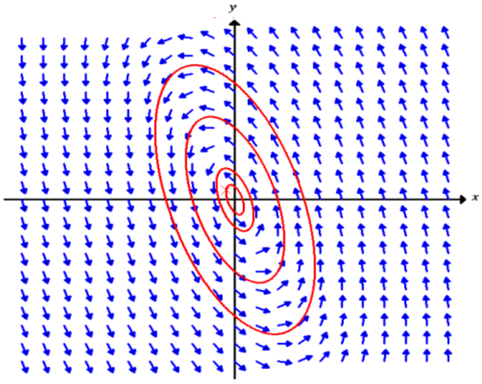

Figure 6. The graph of center to two dimensional dynamical system with (0,0) fixed point

Some physical application

As indicated in Figure 7, let us consider the pendulum that is released from the rest for a rigid body undergoing pure rotation about a pivot axis. The inertia about the pivot O times the angular acceleration is equal to the applied torque.

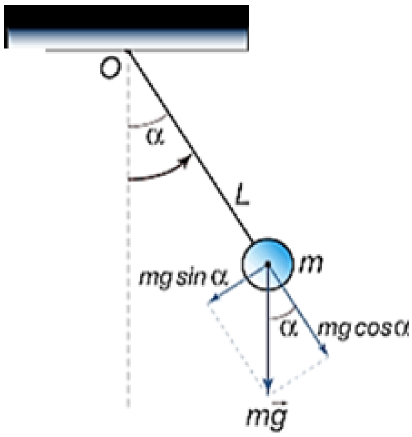

Figure 7. Pendulum

For the pendulum shown in Figure 7, the inertia about O is ml^2, the angle from the vertical is $\alpha\ (t\ )$, the angular acceleration is $\alpha\ ''(t)$, and the down ward gravitational force mg gives a torque of $-mglsinx$. If the air resistance is proportional to the velocity $l\ x'$, say $cl\ x'$, then it gives an additional torque $-\ cl^2x'$, where $\alpha = x$; so, the equation of motion is:

$ml2x'' = -mgl\ sin\ x\ -\ cl2x'or\ x''\ +$

$rx'\ + \frac{g}{l}\ sin\ x\ = 0 \quad \ldots\ldots (\beta)$

Where $r\ \equiv \frac{c}{m}$. The rx' term is damping term. For definiteness, let $\frac{g}{l}$=1 and Consider the following cases.

*Case 1***.** The undamped case where $r = 0$

For small motions we can approximate $sinx$ by the first term of its Taylor series, $sin\ x\ \approx\ x$, so that we have the simple harmonic oscillator equation:

$x''\ +\ x\ =\ 0$ Or $\begin{cases} x' = y \\ y' = -x \end{cases}$ $\quad \ldots\ldots.(\epsilon)$.

The equilibrium point of (ϵ) is $(x, y) = (0,0)$ and its phase portrait is given in the Figure 8.

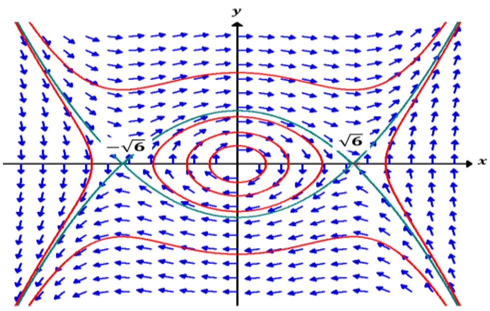

Figure 8. Stable center for damped case

To study larger motions, suppose that we approximate $sin\ x$ by the first two terms of its Taylor series, that is: $sinx \approx x - \frac{x^3}{6}$.Then we have the non-linear, but still approximate, equation of motion $''\ +$ $x\ -\ \frac{1}{6}x^3\ = 0$

The latter is of the same form as the equation governing the rectilinear motion of a mass restrained by “a soft spring” .The system

$$\begin{cases} x' = y \\ y' = -x + \frac{x^3}{6} \end{cases} \quad \ldots\ldots (\gamma)$$

(γ) Has a center at (0, 0) and saddles at $(\pm\sqrt{6}, 0)$ in the $(x,\ y)$ phase plane and its phase portrait is shown in Figure 9.

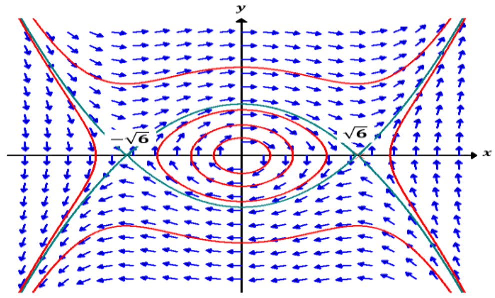

Figure 9. Phase portrait for the improved model (large motion)

Finally, if we keep $sinx$ intact, then we have the full non- linear system:

$$\begin{cases} x' = y \\ y' = -\ \text{sinx} \end{cases} \quad \ldots\ldots(\tau)$$

With singular points at $(x, y)\ =\ (n\pi, 0)$ for $=\ 0, \pm1, \pm2, \ldots.$. To classify these singularities, let us linearize (τ) about the singular point $(x,\ y)\ =\ (n\ \pi\ ,\ 0)$ using linearization techniques. Doing so, knowing that sin n π =0 and $cosn\pi = (-1)^n$ and setting $X = x - n\pi$ and $Y = y - 0 = y$, the linearized version of (τ) is:

$$\begin{cases} X' = Y \\ Y' = (-1)^{n+1}X \end{cases}.$$

Here, we have centers at $x = 0, \pm 2\pi, \pm 4\pi, \ldots.$ and saddles at $x = \pm\pi, \pm 3\pi, \ldots$ on the x axis.

we have centers at x=0,±2p,±4p ,… and saddles at x=±p,±3p,… on the x axis.

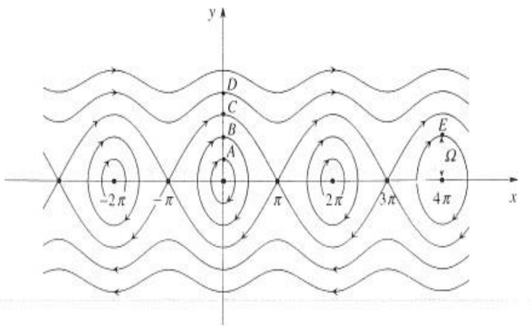

Figure 10. phase portrait of the full nonlinear system $x'' + \sin x = 0$.

To understand the phase portrait as indicated in Figure 10, suppose that the pendulum is hanging straight down ($x = 0$) initially, and that we impart an initial angular velocity $y\,(0)$, so that the initial point is A, B, C, or D in Figure 11. If we start at A, then we follow a closed Orbits that is very close to elliptical, and the motion is very close to simple harmonic motion at frequency $\omega = 1$. If we start at B, the orbit is not so elliptical, there is an increase in the period and the motion deviates somewhat from simple harmonic. If we start at C, then we approach the saddle at $x = \pi$ as $t \to \infty$; that is, the pendulum approaches the inverted position as $t \to \infty$. If we impart more energy by starting at D, then, even though it slows down as it approaches the inverted position, it has enough energy to pass through that position and to keep going round and round indefinitely.

*Case 2***.** Damping case where $r = \frac{c}{m} \neq 0$

With damping, the system is

$$x'' + rx' + \frac{g}{l}sinx = 0,$$

where $r = \frac{c}{m}$, and with small angle approximation, $sinx \approx x$ and the system is $x'' + \frac{c}{m}x' + \frac{g}{l}x = 0.$ Let $x' = y$.Then, we have:

$$\begin{cases} x' = y \\ y' = -\frac{g}{l}x - \frac{c}{m}y \end{cases} \quad \ldots..\ (\phi).$$

The equilibrium point of the system is $(x, y) = (0,0)$.

Linearized equation of the system in the: down position is $\begin{cases} x' = y \\ y' = -\frac{g}{l}x - \frac{c}{m}y \end{cases}$ and up position is $\begin{cases} x' = y \\ y' = \frac{g}{l}x - \frac{c}{m}y \end{cases}$. The corresponding Eigen values are $\lambda_{1,2} = \frac{-\frac{c}{m} \pm \sqrt{(\frac{c}{m})^2 - 4(\frac{g}{l})}}{2}$ and $\lambda_{1,2} = \frac{-\frac{c}{m} \pm \sqrt{(\frac{c}{m})^2 + 4(\frac{g}{l})}}{2}$ respectively.

Now in the down position, we focus on character of fixed point (0,0) in dependence of the values of l, c, g. For their different values, phase portraits of system will be displayed and stability of trivial solution will be determined. There are three cases depending on discriminant part.

I. Critical damped pendulum

Critical damping in this case is occur when the discriminant part is zero. That is,

$c = c_{cr} = 2m\sqrt{\frac{g}{l}}$. Hence value of damping coefficient is equal to value of critical damping coefficient. Since all coefficients must be positive, there exists singe real negative eigenvalues

$\lambda_1 = -\frac{c}{m}$ trivial solution is asymptotically stable, equilibrium point is stable node and the phase portrait is shown in Figure 11.

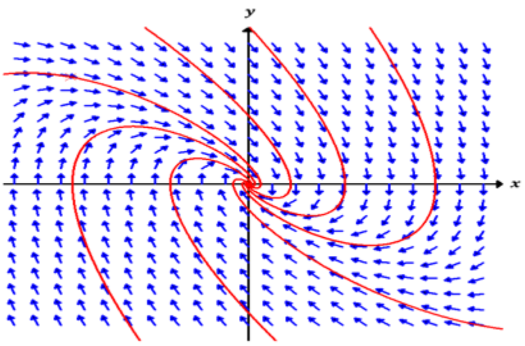

Figure 11. phase portrait of stable node for critical damped pendulum

II. Over damped pendulum

Over damping occurs when the discriminant part is positive. That is $c > c_{cr} = 2m\sqrt{\frac{g}{l}}$. Hence, value of damping coefficient is higher than value of critical damping coefficient. Since all coefficients must be positive, there exists two real negative and different eigenvalues, trivial solution is asymptotically stable, equilibrium point is stable node and the phase portrait is indicated in Figure 12.

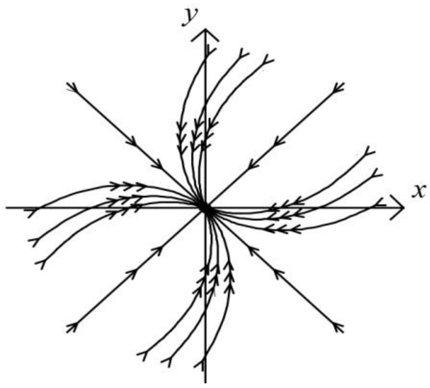

Figure 12. phase portrait of stable node for over damped pendulum

III. Under damped pendulum

In the underdamped pendulum, the discriminant part is negative. That is, $c < c_{cr} = 2m\sqrt{\frac{g}{l}}$

Hence the value of damping coefficient is lower than the value of critical damping coefficient. Since all coefficients must be positive, there exists two eigenvalues which are complex conjugate, their real part is negative, trivial solution is asymptotically stable, equilibrium is stable spiral focus and the phase portrait is displayed in Figure 13.

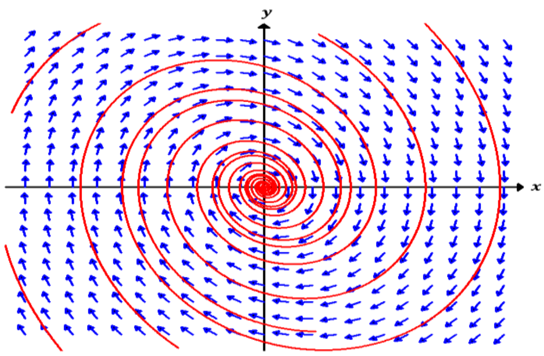

Figure 13. Phase portrait of stable spiral in focus for underdamped pendulum.

Finally, if we retain the entire Taylor series (i.e., if we keep $sinx$ intact), then we have the full non- linear system, with $r = \frac{c}{m}$, or

$$\begin{cases} x' = y \\ y' = -\frac{g}{l}sinx - ry \end{cases} \quad \ldots\ldots (\varphi)$$

With singular points at $(x, y) = (n\pi, 0)$ for $= 0, \pm1, \pm2, \ldots$. To classify these singularities about the singular point $(x,\ y)$ = $(n\pi,\ 0)$, knowing that $sin\, n\pi = 0$ and $cosn\pi = (-1)^n$, then the linearized version has the phase portrait as indicated in Figure 14.

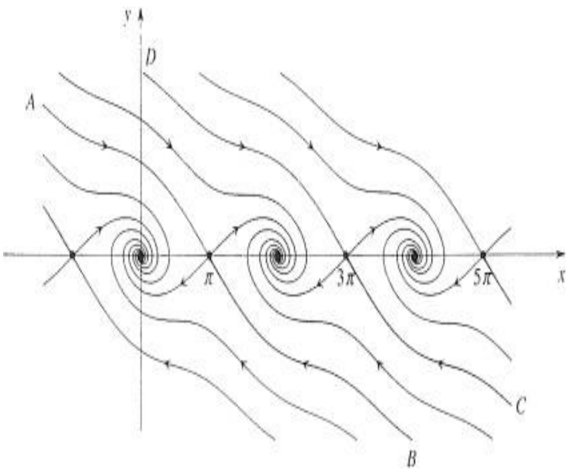

Figure 14. phase portrait for the entire damped pendulum $x'' + rx + \frac{g}{l}sinx = 0$

Conclusion

Non-linear differential equations are incorporated in mathematical models in numerous instances. We have discussed various methods of solving and approximating non-linear differential equations. Due to the difficulty of solving non-linear differential equations, approximation methods have been developed. Qualitative method has been discussed along with several linear and non-linear techniques to approximate or solve the non-linear problems. Singular points, stability, relationship between Portrait diagram and solution of non-linear differential equations were our particular focus with examples of each technique shown. A brief history Taylor's expansion and Jacobean method was discussed and highlighted the recent developments and applications that these expansions provide 2. Visuals were used to demonstrate the accuracy, or lack thereof, of each technique. These techniques are integral in applied mathematics and the correct project allow us to see the behavior of a differential equation when the exact solution may not be attainable. Over all, this paper has demonstrated a few techniques that can be used to approximate non-linear differential equations that can be implemented when the exact solution may not be achievable.

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